Exercise 4.2

Show that $C[0,1]$ is not complete in $\|\cdot\|_2$.

Proof: Put
$$f_n(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{2} - \frac{1}{n}, \\ 1, & \frac{1}{2} + \frac{1}{n} \le x \le 1, \\ \text{linear}, & \frac{1}{2} - \frac{1}{n} < x < \frac{1}{2} + \frac{1}{n}. \end{cases}$$

Then for $m < n$, $\|f_n - f_m\|_2 \le \left(\int_{\frac{1}{2}-\frac{1}{m}}^{\frac{1}{2}+\frac{1}{m}} 1^2 \, dx \right)^{\frac{1}{2}} = \left(\frac{1}{m}\right)^{\frac{1}{2}}$.

Therefore, (f_n) is Cauchy.

Suppose $\|f_n - f\|_2 \to 0$ as $n \to \infty$ for some $f \in C[0,1]$.

Write $f(\frac{1}{2}) = c$.

Then $c \ne 0$ or $c \ne 1$.

If $c \ne 0$, since f is continuous, there exists $n_0 \in \mathbb{N}$ such that $|f(x)| \ge c/2$ for $x \in (\frac{1}{2} - \frac{1}{n_0}, \frac{1}{2} + \frac{1}{n_0})$

For any $n \ge \frac{1}{2n_0}$, $\|f_n - f\|_2 \ge \left(\int_{\frac{1}{2}-\frac{1}{n_0}}^{\frac{1}{2}-\frac{1}{2n_0}} |f_n(x) - f(x)|^2 \right)^{\frac{1}{2}}$
$$= \left(\frac{1}{2n_0} \left(\frac{c}{2}\right)^2 \right)^{\frac{1}{2}}$$
$$= \frac{c}{2\sqrt{2n_0}} > 0.$$

Thus $\|f_n - f\|_2 \not\to 0$ as $n \to \infty$.

Contradiction!

$\square$

Exercise 4.12

Let X and Y be Banach spaces. If $T: X \to Y$ is a bijection, show that $T^*: Y^* \to X^*$ is a bijection. Conclude that if T is an isomorphism, so is T^*.

Proof: (i) Injectivity:

Pick any $y_1^*, y_2^* \in Y^*$ with $y_1^* \neq y_2^*$.

Then $\exists\ y \in Y$ such that $y_1^*(y) \neq y_2^*(y)$.

Since T is surjective, $\exists\ x \in X$ such that $Tx = y$. Then $y_1^*(Tx) \neq y_2^*(Tx)$.

By definition $T^* y_1^*(x) \neq T^* y_2^*(x)$.

Thus $T^* y_1^* \neq T^* y_2^*$

(ii) Surjectivity:

Pick any $x^* \in X^*$.

Put $y^*(y) = x^*(T^{-1}y)$ for any $y \in Y$.

Then $T^* y^*(x) = y^*(Tx) = x^*(T^{-1}Tx) = x^*(x)$ for any $x \in X$.

Therefore, $T^* y^* = x^*$.

(iii) Isomorphism:

Since T is bounded linear, so is T^*.
Since T^* is a bounded linear bijection, by Open Mapping Theorem, $(T^*)^{-1}$ is continuous. Hence, T is an isomorphism.

$\square$